ISSN 1041-9489

SAFETY BRIEF

November 1996 Volume 12, No. 1

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Ladder Slide Out – First Order Analysis

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ABSTRACT

One of the more important collapse modes for straight, combination, and extension ladders is base slide out; the top of the ladder slides down the support wall as the base slips away from it. Various fundamental models have been used to study this behavior. This paper revisits the analytical solutions associated with these models and describes their implications for the analysis, design, and testing of ladders.

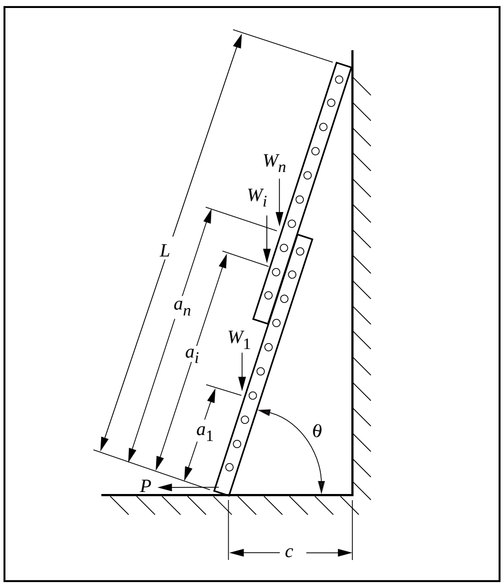


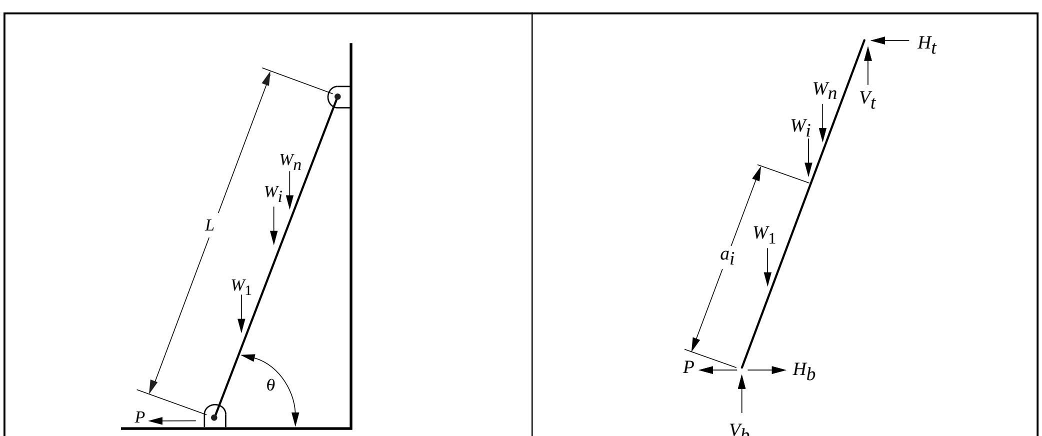
Fig.1 Ladder Geometry and General Loading

I. INTRODUCTION

The geometry and general loading of a typical straight, extension, or combination ladder is shown in Fig. 1 for an angle of inclination θ. In the United States of America, the recommended angle of inclination θ^* corresponds to a ladder set-up with $c/L = 1/4$ or,

$$\theta^* = cos^{-1}\frac{1}{4} = 75.52^\circ \qquad \text{Eq. 1}$$

Several versions of static equilibrium analysis are applied to the ladder in the appendix. The most important finding is the following "no-slip" relationship among the various ladder parameters:



a) Mathematical Model *b) Free Body Diagram*

Fig. 2 Analysis Models

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No-Slip Criterion:

$$\mu_b \geq \frac{(\mu_t + tan\,\theta)\dfrac{P}{W} + \displaystyle\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{(\mu_t + tan\,\theta) - \mu_t \displaystyle\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)} \qquad \text{Eq. 2}$$

where

μ_b ....coefficient of friction between the base and the ladder foot.

μ_t ....coefficient of friction between the wall and the ladder top.

W ....total weight of gravity loading, i.e.,

$$W = \sum_{i=1}^{n} W_i$$

W_i ....i^{th} gravity force.

a_i ....location of the i^{th} gravity force measured from the ladder base.

L ....extended ladder length.

θ ....angle of ladder inclination.

P ....horizontal pulling force – Foot Slip Tests, ANSI A14.2.

The implications of this inequality to ladder technology will be explored in the following sections covering ladder safety analysis and ladder testing.

To arrive at Eq. 2, the physical model of the ladder illustrated in Fig. 1 is idealized by the mathematical model described in Fig. 2a showing a rigid rod hinged at both ends. A free body diagram, Fig. 2b, is then used to establish equilibrium equations. The free body diagram distinguishes between the loading forces, W_i and P, and the respective horizontal and vertical reactions at the top and base of the ladder; H_t, H_b, V_t, and V_b. Table I summarizes the force analyses given in the appendix.

If we designate the center of the gravity forces as $\bar{a}$, then the moment of the total gravity force W about the ladder base is $(W\bar{a})\cos\theta$ and this, by definition, is equal to the sum of the individual moments of gravity forces,

$$\left(\sum_{i=1}^{n} W_i a_i\right)\cos\theta$$

Thus,

$$\bar{a} \equiv \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)a_i \qquad \text{Eq. 3}$$

where

$$W \equiv \sum_{i=1}^{n} W_i$$

Definition $\left(\frac{\bar{a}}{L}\right) \equiv \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)$	**No Incipient Slipping**	**Incipient Slipping**			
		Roller at Ladder Top	Wall First	Simultaneous: Wall and Base	Base First
Equations	*Eq. a*	*Eq. b*	*Eq. c*	*Eq. d*	*Eq. e*
Eq. 4 $V_t =$	$W\left(\frac{\bar{a}}{L}\right) - (H_b - P)\tan\theta$	0	$W\mu_t\left[\frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\theta}\right]$		$\frac{W\left[\left(\frac{\bar{a}}{L}\right) - \mu_b\tan\theta\right] + P\tan\theta}{1 - \mu_b\tan\theta}$
Eq. 5 $V_b =$	$W\left(1 - \frac{\bar{a}}{L}\right) + (H_b - P)\tan\theta$	W	$W\left[\frac{\mu_t\left(1 - \frac{\bar{a}}{L}\right) + \tan\theta}{\mu_t + \tan\theta}\right]$		$\frac{W\left(1 - \frac{\bar{a}}{L}\right) - P\tan\theta}{1 - \mu_b\tan\theta}$
Eq. 6 $H_t =$	$H_b - P$	$\frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\theta}$	$W\left[\frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\theta}\right]$		$\frac{W\mu_b\left(1 - \frac{\bar{a}}{L}\right) - P}{1 - \mu_b\tan\theta}$
Eq. 7 $H_b =$	*Unknown*	$\frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\theta} + P$	$W\left[\frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t + \tan\theta}\right] + P$		$\mu_b\frac{W\left(1 - \frac{\bar{a}}{L}\right) - P\tan\theta}{1 - \mu_b\tan\theta}$
Eq. 8 μ_b:	*Not Applicable*	$\mu_b \geq \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta} + \frac{P}{W}$	$\mu_b \geq \frac{(\mu_t + \tan\theta)\frac{P}{W} + \left(\frac{\bar{a}}{L}\right)}{(\mu_t + \tan\theta) - \mu_t\left(\frac{\bar{a}}{L}\right)}$	$\mu_b = \frac{(\mu_t + \tan\theta)\frac{P}{W} + \left(\frac{\bar{a}}{L}\right)}{(\mu_t + \tan\theta) - \mu_t\left(\frac{\bar{a}}{L}\right)}$	$\mu_b \geq \frac{(\mu_t + \tan\theta)\frac{P}{W} + \left(\frac{\bar{a}}{L}\right)}{(\mu_t + \tan\theta) - \mu_t\left(\frac{\bar{a}}{L}\right)}$ $\mu_b \leq \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta}$

Table I Reaction Forces and Non-Slip Criteria

When there is only one concentrated force on a ladder, n = 1, and $\bar{a} = a_1$ with $W = W_1$. When the center of gravity forces $\bar{a}$ is used in Eq. 2, μ_b takes the forms shown in Eq. 8 in Table I. All of the reaction equations given in Table I are written for a single equivalent concentrated force W located at $\bar{a}$ from the ladder base.

II. LADDER SAFETY ANALYSIS

The various equations presented in the Introduction contain the pull out force P which is associated with ladder testing. This force is not present when climbing a ladder; hence, P will be taken as zero throughout this section. Ladders may collapse by "slide out," by telescoping inward, or by beam-column failure. It will be demonstrated that only the "slide out" problem is under control; telescoping analysis and beam-column analysis require a knowledge of the axial forces on the ladder and this in turn demands that the wall and base reaction forces be known. These reactions, as it turns out, are elusive. On the other hand, bending moments on the ladder can always be determined since it is a simply supported beam.

In the following presentation, both "slide out" variables and ladder reactions will be studied for specific cases.

A. Non-Critical Ladder - No Incipient Sliding

For the ladder illustrated in Fig. 1 there are four unknown reaction forces acting at the top and bottom of the ladder rails; V_t, V_b, H_t, and H_b. Unfortunately, there are only three available equilibrium equations which, by themselves, cannot determine these four unknowns. On the other hand, if one pretends to know one of the reactions, say H_b, then the others can be written in terms containing H_b as shown in Eqs. 4a, 5a, and 6a. Physically, H_b will take on different values every time the ladder is set up and climbed.

For easy visualization, picture a climber of weight W ascending a weightless ladder to location $\bar{a}$ measured from the base. Using $P = 0$ in Table I, one obtains:

$$V_t = W\left(\frac{\bar{a}}{L}\right) - H_b \tan\theta \qquad \text{Eq. 9a}$$

$$V_b = W\left(1 - \frac{\bar{a}}{L}\right) + H_b \tan\theta \qquad \text{Eq. 9b}$$

$$H_t = H_b \qquad \text{Eq. 9c}$$

These equations are also valid for any equivalent force W with a center of loading at $\bar{a}$.

The directions of these reactions are shown in Fig. 2b. They are determined from the physical observations that the wall and base cannot pull on the ladder rails and that sliding will be resisted by upward friction forces at the wall and by inward friction forces at the base. The requirements that $H_b \geq 0$ and $V_t \geq 0$ limit the range of possible values of H_b to

$$0 \leq H_b \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\theta} \qquad \text{Eq. 10}$$

This, in turn, bounds the remaining reactions through Eq. 9; thus,

$$0 \leq V_t \leq W\left(\frac{\bar{a}}{L}\right) \qquad \text{Eq. 11a}$$

$$W\left(1 - \frac{\bar{a}}{L}\right) \leq V_b \leq W \qquad \text{Eq. 11b}$$

$$0 \leq H_t \leq \frac{W\left(\frac{\bar{a}}{L}\right)}{\tan\theta} \qquad \text{Eq. 11c}$$

Axial forces acting on the top and bottom of the ladder rails will be designated A_t and A_b respectively. Each of the reaction forces has axial components that make up A_t and A_b as shown in Fig. 3; thus,

$$A_t = H_t \cos\theta - V_t \sin\theta \qquad \text{Eq. 12a}$$

$$A_b = H_b \cos\theta + V_b \sin\theta \qquad \text{Eq. 12b}$$

The range of values possible for these axial forces may be established using Eqs. 9 and 10; thus,

$$-W \sin\theta\left[\frac{\bar{a}}{L}\right] \leq A_t \leq W \sin\theta\left[\frac{\left(\frac{\bar{a}}{L}\right)}{\tan^2\theta}\right] \qquad \text{Eq. 13}$$

$$W \sin\theta\left[1 - \frac{\bar{a}}{L}\right] \leq A_b \leq W \sin\theta\left[1 + \frac{\left(\frac{\bar{a}}{L}\right)}{\tan^2\theta}\right] \qquad \text{Eq. 14}$$

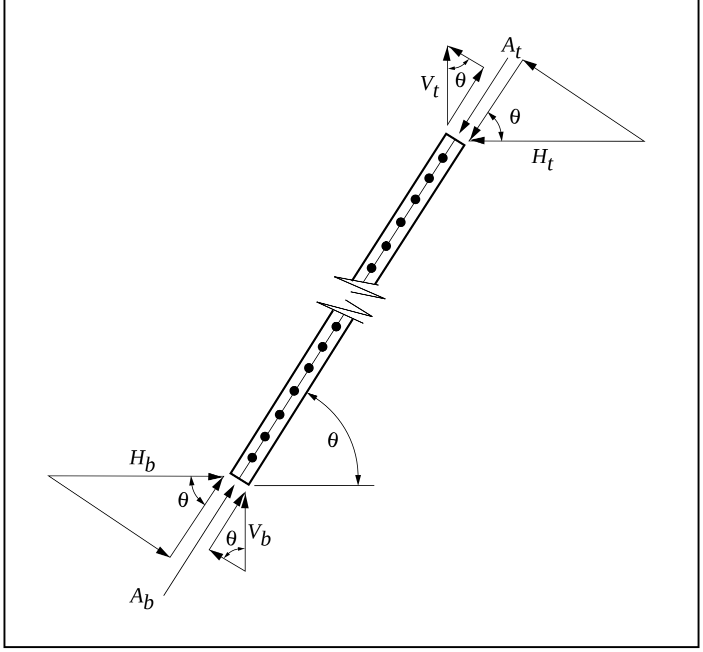


Fig. 3 Axial Rail Forces, A_t and A_b

Observations

1. Most ladder problems fall into the non-critical category since ladders are erected at an inclination in the neighborhood of $\theta^* = 75.52^\circ$. This angle is adopted by ladder manufacturers when determining acceptable levels of friction against sliding.

2. The three available equilibrium equations cannot uniquely determine the four ladder reactions; an additional physical relationship among the reactions is required. Without the associated "fourth equation", one can merely bound the reactions and the axial forces as indicated in Eqs. 10, 11, 13, and 14. The upper and lower bounds are close only when the climber (center of force) is near the bottom of the ladder (at $\bar{a}/L = 0$ they are identical).

3. The maximum achievable values for the reactions and axial forces are given by the upper bounds of the Eqs. 10, 11, 13, and 14 when the climber (center of force) is at the top, i.e., $\bar{a}/L = 1$ and $\theta = \theta^*$:

$$\text{Upper Bounds} \begin{cases} V_t = W \\ V_b = W \\ H_t = 0.258\,W \\ H_b = 0.258\,W \\ A_t = 0.0646\,W & \text{(compression)} \\ A_t = 0.9682\,W & \text{(tension)} \\ A_b = 1.0328\,W & \text{(compression)} \end{cases}$$

When ladder stabilizers are designed to attach to the top of the ladder, the maximum horizontal force $H_t = 0.258W$ and the maximum vertical force $V_t = W$ must be considered. They don't achieve the maximum values simultaneously, maximum V_t is associated with the smallest H_t, i.e., zero (see Eq. 9a).

There is a widespread belief among ladder professionals that the higher one climbs the more critical the ladder becomes. It's true.

4. Because the axial force A_t on the top of a ladder can become tensile, an extension ladder may experience an unlatching force between the fly and base sections. The maximum tension is given by the lower bound in Eq. 13, i.e.,

$$W\left(\frac{\bar{a}}{L}\right) \sin\theta \qquad \text{...max. tension}$$

Using the ladder set-up shown in Fig. 17 of the American National Standard A14.2–1990 (see our Fig. 4), the largest unlatching force occurs when the climber (center of force) stands as high as possible on the base section; $(\bar{a}/L) = (4.5)/13$:

$$W\left(\frac{4.5}{13}\right) \sin 75.52 = 0.335W \qquad \text{(Fly on Top)}$$

If the ladder is flipped over so that the fly section is on the bottom, the climber may stand on the base section with $\bar{a}/L = (7.5)/13$:

$$W\left(\frac{7.5}{13}\right) \sin 75.52 = 0.559W \qquad \text{(Fly on Bottom)}$$

Clearly, it's better to have the fly section on top; however, both values are disturbingly large. If the combined climber and ladder weigh 230 lbs, the "fly on top" gives an $A_t = 77$ lbs., and the "fly on the bottom" gives an $A_t = 129$ lbs. These forces may play a role in developing latch locks or, perhaps, in the unlatching of very flexible extension ladders.

5. Structural engineering would characterize the ladder model described in Fig. 2 as statically indeterminate or hyperstatic to the first degree since equilibrium supplies only three equations for determining four unknown reactions. Structural theory uniquely establishes all four reactions by supplying a fourth equation based on a geometric concept called compatibility. Here, the ladder and hinge supports are treated as flexible entities that must stay attached after being deformed by the gravity and reaction forces. Unfortunately, the flexibility of the top and bottom surfaces cannot be accounted for because of their elusive and fickle nature. Indeed, the reactions will take on different values even when ladder set-ups are nominally identical.

6. If incipient or real sliding takes place at one of the ladder support surfaces, this physical condition produces a fourth relationship among the reactions that allows a complete and unique characterization of V_t, V_b, H_t, H_b, A_t, and A_b. Otherwise they remain random, albeit bounded.

7. The random nature of the reactions may lead to incipient sliding at either the base or the wall. The present analysis gives no indication where it will first occur.

B. Roller Top Ladders (Frictionless Wall)

If the wall is contacted by rollers, no vertical resistance is available to support the top of the ladder. This physical fact provides the "fourth equation" required to uniquely determine the four reaction forces:

$$V_t = 0 \qquad \text{Eq. 15a}$$

$$V_b = W \qquad \text{Eq. 15b}$$

$$H_t = H_b = W \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta} \qquad \text{Eq. 15c}$$

Here, the "no slip" criterion is

$$\mu_b \geq \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta} \qquad \text{...no slip criterion} \qquad \text{Eq. 16}$$

where we observe that the normalized center of force $(\bar{a}/L)$ has been used to simplify the summation expression used throughout the appendix, i.e.,

$$\left(\frac{\bar{a}}{L}\right)=\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)$$

Observations

1. This simple case is the only one where the reaction forces are <u>always known</u> before and at incipient slipping of the base. When slipping occurs, dynamic forces take over the problem.

2. For a weightless ladder that is ascended by a climber of weight W, $(\bar{a}/L)$ may range from zero to unity as the climber moves from the base to the very top. When an unloaded real ladder is considered, its center of gravity is near the middle, $\bar{a}/L \approx 1/2$. When this weight is combined with that of a climber, the center of force of the combined loading can never be exactly at the bottom or exactly at the top even when the climber is all the way at the bottom or top of the ladder. Using the subscript c for the climber and ℓ for the ladder, observe that the center of force may be written as,

$$\left(\frac{\bar{a}}{L}\right)=\left(\frac{W_c}{W}\right)\left(\frac{a_c}{L}\right)+\left(\frac{W_\ell}{W}\right)\left(\frac{a_\ell}{L}\right) \qquad \text{Eq. 17}$$

where

$$0<\left(\frac{a_\ell}{L}\right)<1 \qquad \text{(ladder center of gravity)}$$

$$0\le\left(\frac{a_c}{L}\right)\le 1 \qquad \text{(location of climber)}$$

when

$$\left(\frac{a_c}{L}\right)=0,\left(\frac{\bar{a}}{L}\right)=\left(\frac{W_\ell}{W}\right)\left(\frac{a_\ell}{L}\right)\neq 0 \qquad \text{Eq. 18a}$$

and when

$$\left(\frac{a_c}{L}\right)=1,\left(\frac{\bar{a}}{L}\right)=\frac{W_c}{W}+\left(\frac{W_\ell}{W}\right)\left(\frac{a_\ell}{L}\right)<\frac{W_c}{W}+\frac{W_\ell}{W}=1 \qquad \text{Eq. 18b}$$

Q.E.D.

3. Equation 15c indicates that the largest values for the horizontal reaction forces occur when $(\bar{a}/L)$ is as large as possible. Since it was just established that $(\bar{a}/L)<1$, we may choose unity to provide a conservative upper bound on the reactions; thus,

$$H_t=H_b<\frac{W}{\tan\theta}=0.258W\Big|_{\theta=75.52^\circ}$$

4. To assure that a ladder will not slip during general usage, the base friction μ_b must be selected for the worst case scenario, i.e., for $(\bar{a}/L)$ as large as possible. Using the upper bound on the center of force, $\bar{a}/L=1$, a conservative "no slip" criterion is provided by Eq. 16,

$$\mu_b\ge\frac{1}{\tan\theta}=0.258\Big|_{\theta=75.52^\circ} \qquad \text{Eq. 19}$$

5. Because the "no slip" base friction μ_b given in Eq. 16 does not depend on the magnitude of the applied weights, various climbing tricks are available:

   <u>Trick #1:</u> If the ladder stands under its own weight, it has sufficient friction μ_b to support a climber at least up to the location of the ladder's center of gravity, (a_ℓ/L). Without a climber, Eq. 17 describes the normalized center of force $(\bar{a}/L)$ as

$$\left(\frac{\bar{a}}{L}\right)=\left(\frac{0}{W}\right)\left(\frac{a_c}{L}\right)+\left(\frac{W}{W}\right)\left(\frac{a_\ell}{L}\right)=\left(\frac{a_\ell}{L}\right) \qquad \text{(ladder center of gravity)}$$

   With a climber located at $\left(\frac{a_c}{L}\right)=\left(\frac{a_\ell}{L}\right)$, Eq. 17 shows that

$$\left(\frac{\bar{a}}{L}\right)=\left(\frac{W_c}{W}\right)\left(\frac{a_\ell}{L}\right)+\left(\frac{W_\ell}{W}\right)\left(\frac{a_\ell}{L}\right)=\left(\frac{a_\ell}{L}\right)\left(\frac{W_c+W_\ell}{W}\right)=\left(\frac{a_\ell}{L}\right) \qquad \text{(ladder center of gravity)}$$

   Thus, the center of force is the same and the free standing ladder has proof tested the system up to (a_ℓ/L). No information is provided for climbs above the ladder's center of gravity.

   It should be noted that all falls resulting from ladder "slide out" drop the climber from a height at least equal to $a_\ell \sin\theta$. Not very nice!

   <u>Trick #2:</u> Pulling down on a rope tied to the top of the ladder before climbing, proof tests the ladder and assures that ladder sliding will never occur. Insufficient friction is revealed when the ladder comes crashing down during the pull test (watch your head!).

   <u>Trick #3:</u> When descending a ladder whose "slide out" integrity is unknown, pushing down on the top rung from a stable perch will proof test the ladder set up.

C. First Incipient Slipping at Wall

The assumption of incipient wall slip, $V_t=H_t\mu_t$, provides the "fourth equation" that allows the four reaction forces to be uniquely determined. These are shown in Table I, column c, together with the associated "no slip" criterion.

Observations

1. The three reactions, V_t, H_t, and H_b are all proportional to the center of force location $(\bar{a}/L)$. Consequently, they achieve

their maximum values when $(\bar{a}/L)$ is as large as possible. Choosing $(\bar{a}/L) = 1$ bounds these reactions.

2. In the case of the vertical base reaction, V_b, its maximum value is achieved when the center of force is as small as possible. We have shown that $(\bar{a}/L)$ is never zero; nevertheless, it is useful to bound V_b by taking $(\bar{a}/L) = 0$ in Eq. 5c; thus, $V_b = W$ (upper bound).

3. When $P = 0$, all four reactions are proportional to W, the total gravity loading.

4. The most important finding in this "wall first" case is the expression for the "no slip" criterion. Rearranging the terms given in Eq. 8c for $P = 0$, we obtain,

$$\mu_b \geq \frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t\left[1-\left(\frac{\bar{a}}{L}\right)\right]+\tan\theta} \quad \textit{... no slip criterion} \qquad \text{Eq. 20}$$

This same expression is obtained in every other case studied. Observe that when the wall friction is negligible, $\mu_t = 0$, Eq. 20 reduces to the "Roller Top" case given by Eq. 8b. Indeed, all the reactions described by Eqs. 4c, 5c, 6c, and 7c reduce to those given by Eqs. 4b, 5b, 6b, and 7b for the "Roller Top" case.

5. When choosing a safe base friction for general climbing, the largest possible value for the center of force must be taken. Notice that the numerator in Eq. 20 is largest and the denominator smallest when $(\bar{a}/L)$ is maximized. It is always conservative to take $(\bar{a}/L)$ as unity. This leads to the important result that the wall friction plays no role in the selection of a general climbing "no slip" criterion; note that

$$\mu_t\left[1-\left(\frac{\bar{a}}{L}\right)\right]=0 \qquad \text{when} \quad \left(\frac{\bar{a}}{L}\right)=1.$$

Hence, we obtain

$$\mu_b \geq \frac{1}{\tan\theta} \qquad \textit{...no slip criterion}$$

which is exactly the result shown in Eq. 19 for the "Roller Top" case.

6. The onset of incipient wall slip cannot be detected; consequently, the reactions should be thought of as one possible set of forces. Since they may indeed be realized from time to time, the structural integrity of ladders and any associated appliances must take these reactions into account.

7. In terms of "slip out," heavy ladders are safer than light ones when used in general climbing service. The center of the gravity forces is biased toward the ladder's center of gravity which leads to a smaller $(\bar{a}/L)$ when the climber is at the top. Eq. 20 indicates that a smaller μ_b is required to stabilize a ladder with a reduced $(\bar{a}/L)$. If we increase the weight of a ladder proportionally to αW_ℓ where $\alpha > 1$, Eq. 17 may be used to compare the original and enhanced ladder when the climber is at the top, $a_c/L = 1$. To show that

$$\left(\frac{\bar{a}}{L}\right)_{heavy} = \left(\frac{W_c}{W_c+\alpha W_\ell}\right)+\left(\frac{\alpha W_\ell}{W_c+\alpha W_\ell}\right)\left(\frac{a_\ell}{L}\right) <$$
$$\left(\frac{W_c}{W_c+W_\ell}\right)+\frac{W_\ell}{W_c+W_\ell}\left(\frac{a_\ell}{L}\right)$$
$$=\left(\frac{\bar{a}}{L}\right)_{light},$$

we solve for the α that makes the inequality hold. This turns out after manipulation to be

$$\alpha > 1.$$

This, of course, is our definition of an enhanced ladder.

D. First Incipient Slip at Base

If incipient slipping occurs first at the base, this assumption furnishes the "fourth equation" required to determine the reactions and the "no slip" criterion, i.e., $H_b = V_b\mu_b$. The results are described either by Eqs. A16 and A19 or by those shown in Table I, column e.

Observations

1. The formulas for the "base first" reactions are different than those for "wall first" or "Roller Top."

2. In spite of the differences in the reactions, the "no slip" criterion is identical for the "wall first" and "base first" cases.

3. Why would anyone care whether incipient slipping occurs first at the wall or at the base? If one is interested in the reaction forces, one must care! As a climber ascends a ladder the random reaction forces given in Table I, column (a) may lead to a condition of incipient sliding at either the wall or the base. Whichever is realized first, the associated reaction formulas prevail until "slip out" occurs upon further climbing.

4. Incipient slipping can only occur on stable ladders, i.e., $\mu_b \geq \Omega$. The "base first" model is valid under the condition reflected by Eq. A17 where $P = 0$. Consequently, the range of μ_b where incipient slipping may first appear at the base is

$$\Omega\big|_{P=0} = \frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t\left[1-\left(\frac{\bar{a}}{L}\right)\right]+\tan\theta} \leq \mu_b \leq \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta} \qquad \text{Eq. 21}$$

Observe that when the center of force is at the bottom of the ladder, $(\bar{a}/L) = 0$, both bounds are equal to zero and therefore $\mu_b = 0$. Here, all the reactions disappear except $V_b = W$.

At the other extreme when the center of force is at the top, $(\bar{a}/L) = 1$, both bounds become $(1/\tan\theta)$; hence, $\mu_b = 1/\tan\theta$.

We have proven in the Appendix (Section E) that the "wall first" and "base first" reactions are identical when $\mu_b = \Omega$ as we find here.

The largest differences between the "base first" and "wall first" reactions are found at the maximum difference between the two bounds in Eq. 21. Denoting this difference by Ψ, we find

$$\Psi \equiv \frac{\left(\frac{\bar{a}}{L}\right)}{\tan\theta} - \frac{\left(\frac{\bar{a}}{L}\right)}{\mu_t\left[1-\left(\frac{\bar{a}}{L}\right)\right] + \tan\theta} \qquad \text{Eq. 22}$$

To find the optimum $(\bar{a}/L)$ to maximize Ψ, we set the derivative of Ψ equal to zero; thus,

$$\frac{d\Psi}{d\left(\frac{\bar{a}}{L}\right)} = \frac{1}{\tan\theta} - \frac{\mu_t + \tan\theta}{\left\{\mu_t\left[1-\left(\frac{\bar{a}}{L}\right)\right] + \tan\theta\right\}^2} = 0 \qquad \text{Eq. 23}$$

Solving for $(\bar{a}/L)_{opt}$ we obtain

$$\left(\frac{\bar{a}}{L}\right)_{opt} = 1 + \left(\frac{\tan\theta}{\mu_t}\right) - \sqrt{\left(\frac{\tan\theta}{\mu_t}\right)^2 + \left(\frac{\tan\theta}{\mu_t}\right)} \qquad \text{Eq. 24}$$

Taking an example using $\theta = \theta^*$ and $\mu_t = 0.3$, Eq. 24 becomes,

$$\left(\frac{\bar{a}}{L}\right)_{opt} = 1 + \left(\frac{\tan 75.52}{0.3}\right) - \sqrt{\left(\frac{\tan 75.52}{0.3}\right)^2 + \left(\frac{\tan 75.52}{0.3}\right)} = 0.50933$$

The two bounds in Eq. 21 may be evaluated for $(\bar{a}/L)_{opt}$; hence,

$$\frac{0.50933}{0.3[1 - 0.50933] + \tan 75.52} \le \mu_b \le \frac{0.50933}{\tan 75.52}$$

$$0.12671 \le \mu_b \le 0.13153 \qquad \text{Eq. 25}$$

The maximum difference is small, i.e., $\Psi_{max} = 0.004816$. Taking $(\bar{a}/L) = 0.50933$ and the two bounds on μ_b given in Eq. 25, we may compute the "base first" and "wall first" reaction sets shown in Table II.

Reactions	Lower Bound $\mu_b = 0.12671$	Upper Bound $\mu_b = 0.13153$
$V_t =$	$0.03666\ W$	0
$V_b =$	$0.96334\ W$	W
$H_t =$	$0.12207\ W$	$0.13153\ W$
$H_b =$	$0.12207\ W$	$0.13153\ W$
"Base First"	Eq. A16 or	Eq. A16
"Wall First"	Eq. A12	——

Table II Reaction Forces; $\theta = 75.52°$ *and* $(\bar{a}/L)_{opt} = 0.50933$

One observes from Table II a sizable difference between the upper and lower bounds on V_t; all the other reactions are close.

5. When the base friction μ_b is larger than $[(\bar{a}/L)/\tan\theta]$, incipient slipping cannot first appear at the base. Since μ_b is normally selected for general climbing, safety demands that μ_b exceed $1/\tan\theta$. Under these circumstances, real ladders, which always have $(\bar{a}/L) < 1$, will never experience "base first" incipient slipping.

E. Simultaneous Incipient Slipping at Wall and Base

In addition to the three equilibrium equations, "simultaneity" imposes two more equations; $H_b = V_b\mu_b$ and $V_t = H_t\mu_t$. Too many requirements generally preclude a solution which in the present case means that the four reactions cannot be found. Here, however, a very narrow exception arises when the base friction μ_b is exactly equal to Ω.

Observations

1. When $\mu_b < \Omega$, the ladder is unstable and experiences "slip out."

2. When $\mu_b > \Omega$, the ladder is stable, but simultaneous incipient sliding at the wall and base is impossible.

3. Only when $\mu_b = \Omega$ do we achieve both stability and simultaneous incipient behavior. In this case the reaction forces may be calculated from either the "wall first" or the "base first" equations found in Table I, columns (c) or (e).

4. Simultaneous incipient slipping will occur when a climber on a weightless ladder ascends to the location (a_c/L) given by either Eq. A23 or A25 when $P = 0$; thus,

$$\left(\frac{a_c}{L}\right) = \frac{\mu_b(\mu_t + \tan\theta)}{1 + \mu_b\mu_t} \qquad \text{Eq. 26}$$

Attempts to descend this ladder from the top will result in immediate disaster since μ_b must be greater than Ω for the ladder to be stable when (a_c/L) is greater than the value given by Eq. 26.

III. Ladder Testing – Foot Slip

The stability of ladders relative to "slip out" is established in this country using testing protocols defined by various American National Standards. The relationship between testing and analysis is explored in this section.

Figure 4, taken from ANSI A14.2 – (1982 or 1990), depicts the basic setup and criteria associated with the so called design verification test for foot slip. The test protocol is highly standardized; the test unit is a fully extended 16 foot extension ladder (L = 13ft.) operating on the A surfaces of A-C plywood wall and base panels which are presanded using 320 fine wet / dry sandpaper. A vertical test load (dead load: W_d) is applied on the third highest fly rung and a horizontal pulling force (P) is statically applied at the base as shown in Fig. 4. The standards call for Go/No-Go tests for each of the four ladder types where both W_d and P are specified and where passage requires that no movement across the base exceeds *1/4* inch.

It should be noted that the pulling force P provides a means for imposing a safety factor on the design of a ladder. Extra base friction, over and above that needed for climbing, is required to resist P which may be set by rule makers to account for variability in friction measurements, static vs. dynamic friction, differences in operating surfaces, errors in choosing the inclination angle θ and other types of ladder misuse. The usefulness of P as a safety parameter derives from the strict standardization of the testing protocol. Observe that a minimum value of μ_b may be associated with any value P through the strict equality of Eq. 2 which is rewritten as,

$$\mu_b = \frac{(\mu_t + \tan\theta)\dfrac{P}{W} + \left(\dfrac{\bar{a}}{L}\right)}{\mu_t\left[1 - \left(\dfrac{\bar{a}}{L}\right)\right] + \tan\theta} \qquad \text{Eq. 27}$$

The minimum coefficient of base friction is a more useful concept than P for designers who must choose a satisfactory friction material for the ladder foot pads.

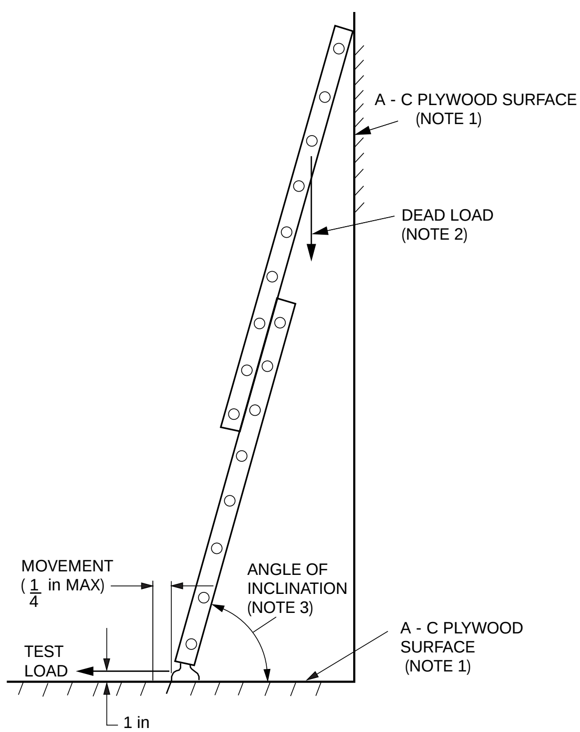
Another important expression of a safety factor is the minimum inclination angle that a ladder may be safely erected. This angle may also be established using Eq. 27. To illustrate these various ideas, an example has been selected that involves four different ladder types of 16-foot aluminum extension ladders of uniform width and equal length sections. Their characteristics are tabulated in Table III in lines 1 through 5 where different weights are associated with the base and fly sections and where their normalized centers of gravity are located at $(a_1/L) = 4/13$ and $(a_2/L) = 9/13$ respectively. In each case considered, the test load W_d shown in lines 9 and 13 of Table III is located 2.5 feet from the top of the ladder or at $(a_d/L) = (11.5)/13$. Equation 3 is used to compute the center of force $(\bar{a}/L)$ associated with the footslip test, e.g.,

$$\left(\frac{\bar{a}}{L}\right) = \left(\frac{W_d}{W}\right)\left(\frac{a_d}{L}\right) + \left(\frac{W_1}{W}\right)\left(\frac{a_1}{L}\right) + \left(\frac{W_2}{W}\right)\left(\frac{a_2}{L}\right) \qquad \text{Eq. 28}$$

$$= \left(\frac{300}{328}\right)\left(\frac{11.5}{13}\right) + \left(\frac{15}{328}\right)\left(\frac{4}{13}\right) + \left(\frac{13}{328}\right)\left(\frac{9}{13}\right)$$

$$= 0.85061 \quad \text{(Type I A - line 7, Table III)}$$

Using the W_d and P given by ANSI A14.2 –1982, lines 9 and 10 in Table III, together with $\mu_t = 0.3$ and $\theta = 75.52^\circ$, the associated base friction becomes:



NOTES:

(1) The grain of the plywood shall be parallel to the direction of loading; the grain on the vertical sheet under the upper end of the fly section shall run in a vertical direction, and the grain on the horizontal sheet under the base section shall be parallel to the direction of the test load.

(2) The dead load shall be applied on the third highest rung.

(3) The angle of inclination shall be 75-1/2º, except that for combination ladders in the extension-ladder orientation a slightly modified angle shall be used so that the tread portions of the steps are horizontal (level).

ANSI A14.2 – 1982

Duty Rating and Type	Test Load* (pounds)	Horizontal Pulling Force (pounds)
Extra heavy duty – Type IA	300	50
Heavy duty – Type I	250	50
Medium duty – Type II	225	50
Light duty – Type III	200	50

* This load is applied on the third highest fly rung.

ANSI A14.2 – 1990

Duty Rating and Type	Test Load* (pounds)	Horizontal Pulling Force (pounds)
Extra heavy duty – Type IA	300	14
Heavy duty – Type I	250	18
Medium duty – Type II	225	20
Light duty – Type III	200	22

* This load is applied on the third highest fly rung.

Fig. 4 ANSI A14.2 Foot Slip Test

Duty Rating Type	Extra Heavy Duty Type I A	Heavy Duty Type I	Medium Duty Type II	Light Duty Type III
1. Base Weight: W_1 (lbs.)	15	14	13	12
2. Fly Weight: W_2 (lbs.)	13	12	11	10
3. Ladder Weight (lbs.)	28	26	24	22
4. (a_1/L)	4/13	4/13	4/13	4/13
5. (a_2/L)	9/13	9/13	9/13	9/13
6. (a_d/L)	11.5/13	11.5/13	11.5/13	11.5/13
7. Foot Slip Test: $(\bar{a}/L)$	0.85061	0.84699	0.84600	0.84477
8. Critical Climb: $(\bar{a}/L)$	0.93691	0.94078	0.94471	0.94872
ANSI A14.2 – 1982				
9. Test Load: W_d (lbs.)	300	250	225	200
10. Pulling Force: P (lbs.)	50	50	50	50
11. Base Friction: μ_b	0.3795	0.4091	0.4297	0.4554
12. Minimum Safe θ	67.80°	66.34°	65.38°	64.19°
ANSI A14.2 – 1990				
13. Test Load: W_d (lbs.)	300	250	225	200
14. Pulling Force: P (lbs.)	14	18	20	22
15. Base Friction: μ_b	0.2626	0.2856	0.3014	0.3211
16. Minimum Safe θ	74.26°	73.03°	72.22°	71.21°

Table III Aluminum Extension Ladders – Length: 16 feet

$$\mu_b = \frac{(0.3 + \tan 75.52)\dfrac{50}{328} + 0.85061}{0.3(1 - 0.85061) + \tan 75.52}$$

$= 0.3795$ (Type I A - line 11, Table III)

This value of μ_b may now be used to establish the minimum safe ladder angle for a 200 lb. climber critically located at the top of the ladder, $(a_c/L) = 1$. The associated $(\bar{a}/L)$ for the climbing scenario is calculated from,

$$\left(\frac{\bar{a}}{L}\right) = \left(\frac{W_c}{W}\right)(1) + \left(\frac{W_1}{W}\right)\left(\frac{a_1}{L}\right) + \left(\frac{W_2}{W}\right)\left(\frac{a_2}{L}\right) \qquad \text{Eq. 29}$$

$$= \frac{200}{228} + \left(\frac{15}{228}\right)\left(\frac{4}{13}\right) + \left(\frac{13}{228}\right)\left(\frac{9}{13}\right)$$

$= 0.93691$ (Type I A - line 8, Table III)

Solving Eq. 27 for θ when $P = 0$, we obtain,

$$Min.\ Safe\ \theta = \tan^{-1}\left\{\frac{\left(\frac{\bar{a}}{L}\right)(1 + \mu_b\ \mu_t)}{\mu_b} - \mu_t\right\} \qquad \text{Eq. 30}$$

$$= \tan^{-1}\left\{\frac{0.93691[1 + (0.3795)(0.3)]}{0.3795} - 0.3\right\}$$

$= 67.80^o$ (Type I A - line 12, Table III)

All of the parameters, code requirements, and calculations are summarized and tabulated in Table III.

Wall Friction μ_t	Stable Base Friction μ_b	Error = $\frac{\mu_b - 0.2408}{0.2408}$
0	0.2420	0.509%
.1	0.2416	0.342%
.2	0.2412	0.176%
.3	0.2408	Assumption
.4	0.2404	-0.156%
.5	0.2400	-0.322%
.6	0.2396	-0.488%
.7	0.2392	-0.654%
.8	0.2388	-0.821%
.9	0.2385	-0.945%
1.0	0.2381	-1.111%

$$\mu_b = \frac{0.93691}{\mu_t\left[1-0.93691\right]+tan\,75.52}$$

Table IV Effect of Wall Friction on Climbing Stability (Type IA, W_c = 200 lbs., ($\bar{a}/L$) = 0.93691, θ = 75.52º)

Wall Friction μ_t	Computed Base Friction μ_b	Error = $\frac{\mu_b - 0.3211}{0.3211}$
0	0.3173	-1.175%
.1	0.3185	-0.801%
.2	0.3198	-0.397%
.3	0.3211	Assumption
.4	0.3223	0.382%
.5	0.3236	0.787%
.6	0.3248	1.161%
.7	0.3260	1.534%
.8	0.3272	1.908%
.9	0.3284	2.282%
1.0	0.3296	2.656%

$$\mu_b = \frac{\left(\mu_t + tan\,75.52\right)\frac{22}{222} + 0.84477}{\mu_t\left[1-0.84477\right]+tan\,75.52}$$

Table V Effect of Wall Friction on Computed Base Friction (ANSI A14.2 – 1990, Type III, ($\bar{a}/L$) = 0.84477, θ = 75.52º)

Observations

1. The friction levels required to just satisfy the ANSI A14.2–1982 standard are found on line 11, Table III (0.3795, 0.4091, 0.4297, 0.4554). These friction coefficients are easy to obtain on most surfaces; almost all footwear will achieve friction levels of 0.5 or greater. Associated with these minimum friction values are the “minimum safe angles of inclination” shown on line 12, Table III (67.80º, 66.34º, 65.38º, 64.19º).

   These angles are very forgiving when compared to the manufacturer's instructions and the ladder safety standards, $\theta = 75.52^\circ$. A 1977 study by Liberty Mutual Insurance Company [1] revealed that the preferred mean angle of ladder inclination among homeowners and carpenters was 73º for 24-foot, 32-foot, and 40-foot aluminum extension ladders. These studies showed a deviation for a 16-foot ladder; the mean was 68.7º which is still above our highest minimum safe angle of 67.8º. The twenty two participants in the study were simply trying to use the most appropriate and comfortable angle for performing a painting task; they had no knowledge of the ANSI recommendation of 75.52° or of any standardized method for determining the angle.

2. It may be observed from line 14, Table III that ANSI A14.2 – 1990 calls for much lower pull forces P than in 1982. This silliness is the result of a printing error in the 1990 standard; the pull criterion never changed [4]. This paper, nevertheless, explores the consequences of the erroneous low level pull forces to illustrate how they lead to very small safety factors on the minimum safe inclination angles. The fictitious pull forces are shown on line 14, Table III. The associated base friction coefficients given in line 15 (0.2626, 0.2856, 0.3014, 0.3211) are very modest and lead to the minimum safe angles shown on line 16 (74.26º, 73.03º, 72.22º, 71.21º). The minimum safe angles are very close to 75.52°. They are unforgiving and demand great precision in setting the recommended angle of inclination.

3. Can ladders be safely set up without recourse to standardized methods for choosing the inclination angle? This question is equivalent to “can a safe μ_b be selected when intuition is used to establish the ladder angle?” According to the findings of Ref. [1], for 16-foot aluminum extension ladders the intuitive methods of homeowners gave rise to a mean setup angle of 68.25º with a standard deviation sigma of σ = 3.86º. Assuming a normal frequency distribution, 99.7% of the angles θ selected by intuition will fall in the range,

   $$Mean - 3\,\sigma < \theta \leq Mean + 3\,\sigma$$

   or,

   $$56.67^\circ \leq \theta \leq 79.83^\circ$$

   Using a Type III aluminum extension ladder, the required safe μ_b at the lower limit for a 200 lb. climber may be found from Eq. 27 with $P = 0, (\bar{a}/L) = 0.94872$ from line 8, Table III and $\mu_t = 0.3$; thus,

$$\mu_b = \frac{0.94872}{0.3(1 - 0.94872) + tan\,56.67^\circ}$$
$$= 0.6177$$

To be conservative, this value of μ_b should be treated as sliding friction; static friction may easily be 15% greater. On this basis the foot pad material must be selected to give

$$\mu_b = 1.15 \text{ X } (0.6177) = 0.7104 \text{ (static)}$$

on arbitrary surfaces where a ladder may be erected. These surfaces may, of course, be dirty, wet, or oily.

At present, no pad materials exist that will meet this demand where intuition prevails in the selection of θ. On the other hand, when $\theta = 75.52^\circ$ is used, the safe μ_b under the preceeding assumptions is only $\mu_b = 0.28$.

4. What is the role of wall friction on the stability of ladders? Throughout this paper it has been assumed in all computations that $\mu_t = 0.3$. It will now be shown that stability is relatively insensitive to μ_t.

   Under general climbing scenarios $(\bar{a}/L)$ must be maximized to produce the minimum required μ_b. Referring to Eq. 27 when $P = 0$, we observe that as $(\bar{a}/L)$ approaches unity, μ_t is multiplied by the bracketed quantity which approaches zero. Consider a Type I A aluminum extension ladder supporting a 200 lb. climber located at the top of the ladder. Here the critical $(\bar{a}/L) = 0.93691$ is shown in line 8, Table III. Using this value and $\theta = 75.52^\circ$, μ_b is calculated for values of μ_t between zero and unity and the results are tabulated in Table IV. As shown in this table, when the wall friction is assumed to be $\mu_t = 0.3$ the error in determining a safe μ_b is less than 1%. The influence of wall friction is seen to be *de minimus*.

5. Despite the fact that the ANSI standard outlines a Go/No-Go test for slip out, the procedure may be made quantitative [2] by continuously increasing P until sliding begins. The maximum pulling force P_{max} is substituted into Eq. 27 to determine the base friction μ_b. The wall friction may be independently measured using the contact hardware at the top of the ladder and the wall surface material.

   Very little is gained by measuring μ_t since it may be approximated as $\mu_t = 0.3$ without exerting much influence on the determination of the base friction. Taking, for example, a 22 lb., Type III, aluminum extension ladder under the standard foot slip setup conditions, we can calculate μ_b using Eq. 27 for various values of μ_t in the range of zero to unity. Adopting the ANSI A14.2 – 1990 protocol where $(\bar{a}/L) = 0.84477$ from line 7, Table III, the values of μ_b are tabulated in Table V. It may be observed that μ_b varies less than 1% for μ_t in the neighborhood of $\mu_t = 0.3$. In the view of this relative insensitivity, it may not be necessary to be so exacting in the specification of wall materials for the ANSI foot slip test protocol.

APPENDIX

A. Equilibrium

When a ladder is stationary the forces shown in the free body diagram depicted in Fig. 2b can be related by three equilibrium equations as follows:

Moment Equilibrium About Ladder Base:

$$H_t L \sin\theta + V_t L \cos\theta - \sum_{i=1}^{n} W_i a_i \cos\theta = 0 \qquad \text{Eq. A1}$$

Vertical Equilibrium:

$$V_t + V_b - \sum_{i=1}^{n} W_i = 0 \qquad \text{Eq. A2}$$

Horizontal Equilibrium:

$$H_t - H_b + P = 0 \qquad \text{Eq. A3}$$

These equations may be solved for V_t, V_b, and H_t in terms containing only the unknown reaction H_b; thus,

$$V_t = \sum_{i=1}^{n} W_i \left(\frac{a_i}{L}\right) - H_b \tan\theta + P \tan\theta \qquad \text{Eq. A4}$$

$$V_b = \sum_{i=1}^{n} W_i \left(1 - \frac{a_i}{L}\right) + H_b \tan\theta - P \tan\theta \qquad \text{Eq. A5}$$

$$H_t = H_b - P \qquad \text{Eq. A6}$$

B. Model Validity

The ladder support surfaces can only resist compression; if tension is required to maintain equilibrium the ladder would lift off the supports. Consequently, the model assumed in Fig. 2 is invalid if $H_t < 0$ or $V_b < 0$.

The shear reactions V_t and H_b are developed through friction which always resists incipient or real motion. During slide out, the directions of frictional resistance shown in Fig. 2b for V_t and H_b must be achieved. Any assumptions or conditions that lead to negative values for these reactions will invalidate the model; thus, validity demands that $V_t \geq 0$ and $H_b \geq 0$.

C. Top Roller (Frictionless Wall)

To minimize scratching of the support walls, ladders sometimes incorporate rollers into the top of the siderails. This case is a favorite textbook example modelled as a frictionless wall, $\mu_t = 0$, where no vertical reaction can be sustained, i.e.,

$$V_t = 0 \qquad \text{Eq. A7}$$

This equation taken together with the three equilibrium equations, A4, A5, and A6, uniquely determine the four reactions:

$$V_t = 0 \qquad \text{Eq. A8a}$$

$$V_b = \sum_{i=1}^{n} W_i \equiv W \qquad \text{Eq. A8b}$$

$$H_b = \frac{W}{\tan\theta} \sum_{i=1}^{n} \left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) + P \qquad \text{Eq. A8c}$$

$$H_t = \frac{W}{\tan\theta} \sum_{i=1}^{n} \left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) \qquad \text{Eq. A8d}$$

As long as the ladder remains stationary these reaction forces remain valid.

The ladder will cease to be in equilibrium when the horizontal base reaction H_b exceeds the frictional resistance of the ladder feet given by the product of the normal base reaction V_b and the coefficient of friction between the floor surface and the ladder feet, μ_b. Consequently, a "no slip" criterion becomes,

$$H_b \leq V_b \mu_b \qquad \text{...no slip criterion} \qquad \text{Eq. A9}$$

Substituting from Eqs. A8b and A8c, we obtain

$$\frac{W}{\tan\theta} \sum_{i=1}^{n} \left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) + P \leq W \mu_b$$

or,

$$\mu_b \geq \frac{1}{\tan\theta} \sum_{i=1}^{n} \left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) + \frac{P}{W} \qquad \text{...no slip criterion} \qquad \text{Eq. A10}$$

It may be noted that the following normalized and dimensionless expression gives the i^{th} gravity force as a fraction of the total gravity force W,

$$\left(\frac{W_i}{W}\right), \qquad i = 1,2,...n$$

The expression,

$$\left(\frac{a_i}{L}\right), \qquad i = 1,2,...n$$

is the normalized and dimensionless location of the i^{th} gravity force W_i; it is expressed as a fraction of the ladder length measured from the base. The set (W_i, a_i) gives the load distribution on the ladder. When $P = 0$, the case of ordinary climbing, we observe that Eq. A10 indicates that the μ_b required for stability is independent of the weight on the ladder; it depends only on the weight distribution.

D. First Incipient Slipping at Wall

If it is assumed that the incipient slipping appears first at the wall, we may write the vertical reaction V_t as the product of the normal wall force H_t and the coefficient of friction μ_t between the wall and the top of the ladder. With this "fourth equation", the four reaction forces are uniquely determined using Eqs. A4, A5, and A6; thus,

$$V_t = H_t \mu_t \qquad \text{Eq. A11}$$

or,

$$\sum_{i=1}^{n} W_i \left(\frac{a_i}{L}\right) - H_b \tan\theta + P \tan\theta = \left(H_b - P\right)\mu_t$$

Manipulation provides,

$$H_b = P + W \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\left(\mu_t \tan\theta\right)} \qquad \text{Eq. A12a}$$

$$H_t = W \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\left(\mu_t \tan\theta\right)} \qquad \text{Eq. A12b}$$

$$V_b = W \frac{\mu_t \left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] + \tan\theta}{\left(\mu_t + \tan\theta\right)} \qquad \text{Eq. A12c}$$

$$V_t = W\mu_t \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\left(\mu_t + \tan\theta\right)} \qquad \text{Eq. A12d}$$

These reaction forces prevail until the base slides, i.e., $H_b > V_b\, \mu_b$. Consequently, the "no slip" criterion becomes,

$$H_b \le V_b \mu_b \qquad \text{...no slip criterion} \qquad \text{Eq. A13}$$

or,

$$P + W \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\left(\mu_t + \tan\theta\right)} \le \mu_b W \frac{\mu_t \left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] + \tan\theta}{\left(\mu_t + \tan\theta\right)}$$

Thus,

$$\mu_b \ge \frac{\left(\mu_t + \tan\theta\right)\frac{P}{W} + \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\mu_t \left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] + \tan\theta} \equiv \Omega \qquad \text{Eq. A14}$$

...no slip criterion

E. First Incipient Slipping at Base

If the base is first to achieve incipient sliding, the "fourth equation" to be used in conjunction with the three equilibrium equations to uniquely establish the four reaction forces is

$$H_b = V_b \mu_b \,. \qquad \text{Eq. A15}$$

Using Eq. A5 this becomes, after rearranging,

$$H_b\left(1 - \mu_b \tan\theta\right) = \left[\sum_{i=1}^{n} W_i \left(1 - \frac{a_i}{L}\right) - P \tan\theta\right]\mu_b$$

If $\left(1 - \mu_b \tan\theta\right) \ne 0$,

$$H_b = \mu_b \frac{W\left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] - P\tan\theta}{\left(1 - \mu_b \tan\theta\right)} \qquad \text{Eq. A16a}$$

$$H_t = \frac{W\mu_b\left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] - P}{\left(1 - \mu_b \tan\theta\right)} \qquad \text{Eq. A16b}$$

$$V_b = \frac{W\left[1 - \sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right] - P\tan\theta}{\left(1 - \mu_b \tan\theta\right)} \qquad \text{Eq. A16c}$$

$$V_t = \frac{W\left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) - \mu_b \tan\theta\right] + P\tan\theta}{\left(1 - \mu_b \tan\theta\right)} \qquad \text{Eq. A16d}$$

The validity of the "base first" model, when $P = 0$, requires that these reactions be nonnegative. The numerators in the formulas for H_b, H_t, and V_b are all nonnegative; the denominators will be positive if $(1 - \mu_b \tan\theta) > 0$. If V_t is to be nonnegative, the quantity in brackets shown in Eq. A16d must be nonnegative, i.e.,

$$\left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) - \mu_b \tan\theta\right] \ge 0$$

This equation implies that $\left(1 - \mu_b \tan\theta\right)$ will always be positive. Observe that for real ladders,

$$\left(1 - \mu_b \tan\theta\right) > \left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right) - \mu_b \tan\theta\right] \ge 0$$

Thus, the model will be valid if and only if

$$\mu_b \le \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\tan\theta} < \frac{1}{\tan\theta} \qquad \text{Eq. A17}$$

For the ladder to begin sliding the top must slip downward which implies that the V_t exceeds the frictional resistance $H_t \mu_t$. The "no slip" criterion becomes

$$V_t \le H_t \mu_t \qquad \text{...no slip criterion} \qquad \text{Eq. A18}$$

Substituting from Eq. A16 gives,

$$\frac{W\left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)-\mu_b \tan\theta\right]+P\tan\theta}{\left(1-\mu_b \tan\theta\right)} \le \mu_t \frac{W\mu_b\left[1-\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right]-P}{\left(1-\mu_b \tan\theta\right)}$$

Because $\left(1-\mu_b \tan\theta\right) > 0$,

$$\mu_b \ge \Omega \qquad \textit{...no slip criterion} \qquad \text{Eq. A19}$$

We observe that Eq. A19 is identical to the "no slip" criterion, Eq. A14, where incipient slipping begins at the wall. On the other hand, the "base first" reactions given by Eq. A16 are different than the "wall first" reactions described by Eq. A12. When the equality sign holds in Eqs. A14 and A19, the reactions are identical in the "base first" and "wall first" cases. This may readily be seen by equating any of the reactions given by Eqs. A12 and A16 and solving for μ_b. For example, using V_t we obtain,

$$W\mu_t \frac{\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)}{\left(\mu_t + \tan\theta\right)} = \frac{W\left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)-\mu_b \tan\theta\right]+P\tan\theta}{1-\mu_b \tan\theta}$$

Eq. A20a

Solving for μ_b gives,

$$\mu_b = \Omega \qquad \text{Eq. A20b}$$

F. Simultaneous Incipient Slipping at Wall and Base

This case rounds out our study of "slip out" since we have progressed from too little information to just the right amount and now to too much information. Simultaneous incipient slipping at the wall and base produces two equations in addition to the three from equilibrium; thus, we have five equations to determine four unknown reactions. In general, this implies that a solution for the reaction forces does not exist. There is one exception however, and this is demonstrated in the following development.

Because incipient slipping takes place at the wall, $V_t = H_t\mu_t$. Consequently, the reactions must be the same as those developed for the "wall first" case, Eq. A12. Now, we add the fact that incipient slipping at the base occurs simultaneously, $H_b = V_b\mu_b$. Substituting H_b and V_b from Eqs. A12a and A12c into this equation we obtain,

$$\mu_b = \Omega \qquad \text{Eq. A21}$$

Note that in the "wall first" case an identical substitution was made, albeit for a different reason, using the inequality $H_b \ge V_b\ \mu_b$ in contrast to the present case using the equal sign. The strict equality of Eq. A21 indicates that simultaneous achievement of incipient slip at the wall and base cannot occur except in one special case where the loading produces a critical value of the center of gravity forces. Solving Eq. A21 we find

$$\left[\sum_{i=1}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right]_{critical} = \frac{\left(\mu_b - \frac{P}{W}\right)\left(\mu_b + \tan\theta\right)}{1+\mu_b\mu_t} \qquad \text{Eq. A22}$$

When all the gravity loads are known except for the location a_c of a single climber of weight W_c, we may write the bracketed quantity as

$$\left(\frac{W_c}{W}\right)\left(\frac{a_c}{L}\right) + \sum_{\substack{i=1\\ i\ne c}}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)$$

Using this representation, Eq. A22 may be solved for the critical location (a_c/L),

$$\left(\frac{a_c}{L}\right) = \left[\frac{1}{\left(\frac{W_c}{W}\right)}\right]\left[\frac{\left(\mu_b - \frac{P}{W}\right)\left(\mu_t + \tan\theta\right)}{1+\mu_b\mu_t} - \sum_{\substack{i=1\\ i\ne c}}^{n}\left(\frac{W_i}{W}\right)\left(\frac{a_i}{L}\right)\right]$$

Eq.A23

Another approach to the case of simultaneous incipient slipping is described by Timoshenko [3] who uses a graphic method that takes advantage of the known directions of the wall and base reactions at incipient slip. Referring to Fig. 5, the so called friction angles α and β are defined as

$$\tan\alpha = \mu_b \qquad \text{Eq. A24a}$$

$$\tan\beta = \mu_t \qquad \text{Eq. A24b}$$

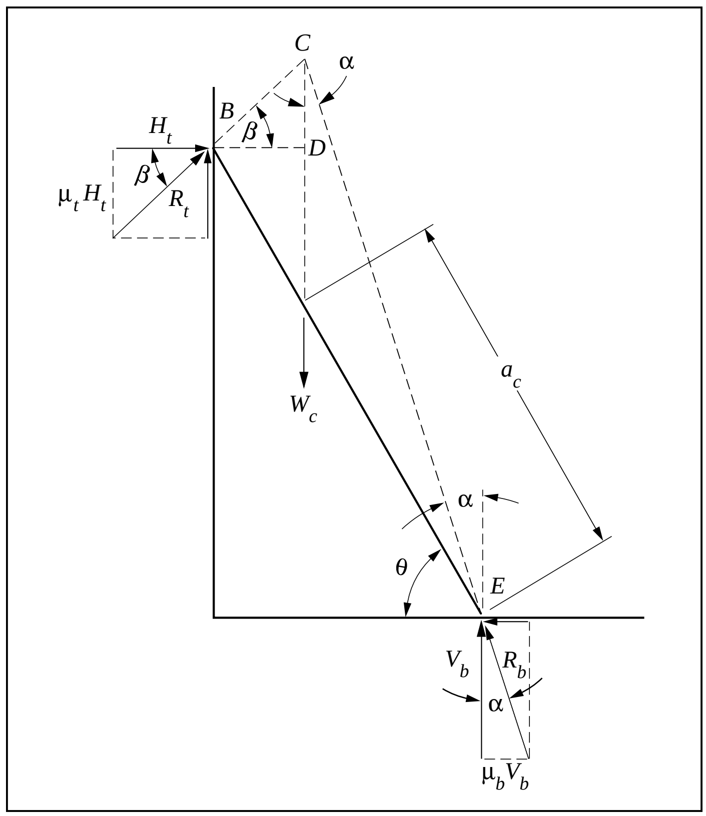


Fig. 5 Timoshenko's Graphic Solution

At incipient slip, these angles define the orientation of the respective base and wall reaction resultants, R_b and R_t. From statics we know that equilibrium requires that three nonparallel forces pass through a single point; in Fig. 5 point C is the intersection of the load path for W_c and the resultant paths for R_b and R_t. This, of course, immediately defines a unique a_c. If the drawing in Fig. 5 is constructed to scale, a_c may be measured directly. Otherwise a_c may be computed using the following relationships:

From geometry,

$$\overline{BD} = \overline{BC}\, cos\, \beta$$

$$\overline{BD} = L\, cos\, \theta - a_c\, cos\, \theta$$

From the Law of Sines (triangle BCE),

$$\frac{\overline{BC}}{sin\left[90 - (\theta + \alpha)\right]} = \frac{L}{sin\left[90 - \beta + \alpha\right]}$$

Solving these three equations gives,

$$\left(\frac{a_c}{L}\right) = \left[1 - \frac{cos(\alpha + \theta)\, cos\, \beta}{cos(\alpha - \beta)\, cos\, \theta}\right] \qquad \text{Eq. A25}$$

The following observations are useful:

1. When the base is frictionless, $\alpha = 0$, $(a_c/L) = 0$ and a climber cannot ascend.

2. If the ladder angle is $\theta = 90 - \alpha$, the base reaction resultant R_b lines up with the side rails and $(a_c/L) = 1$ allowing the climber to ascend to the top of the ladder.

3. If the definitions of α and β from Eq. A24 are substituted into Eq. A25, one recaptures the expression given by Eq. A22 for the center of force when $P = 0$.

REFERENCES

1. Irmin, Charles H. and Marian Vejvodea. 1977. "An Investigation of the Angle of Inclination for Setting Non-self-supporting Ladders." *Professional Safety* 22, no. 7 (July): pp. 34-39.

2. Brickman, Dennis B. and Ralph L. Barnett. 1993. "Quantification versus Go/No-Go Criteria," *Reliability, Stress Analysis, and Failure Prevention –1993*, DE - 55. New York: American Society of Mechanical Engineers, pp. 9-15.

3. Timoshenko, S. and D. H. Young.1951. *Engineering Mechanics, 3rd ed*. New York: McGraw-Hill Book Co., Inc., pp. 63-64.

4. Errata Sheet, ANSI A14.2–1990, 9/19/91.

SAFETY BRIEF

November 1996 – Volume 12, No. 1

Editor: Paula L. Barnett

Illustrated and Produced by Triodyne Graphic Communications Group